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On the Second Twist Number

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Abstract

It has been shown that the twist number of a reduced alternating knot can be determined by summing certain coefficients in the Jones Polynomial. In the discovery of this twist number, it became evident that there exist higher order twist numbers which are the sums of other coefficients. Some relations between the second twist number and the first are explored while noting special characteristics of the second twist number.

1 Background Information

We begin this paper with a short review of knot theory and some important definitions. Mathematically, a *knot* is a *closed* curve in space. Further, a *link* is a collection of one or more interlinked knots. Here we see that a knot is just a link with one component. The simplest link is just a circle or the *unknot*, since it has no crossings. If we consider a link with just one crossing, we realize that this is really the unknot again with a kink in it. Now in considering a link with two crossings we have a choice: either we can get an unknot with two kinks in it or we can get a link composed of two unknots as in figure 1, called the Hopf link. The simplest nontrivial knot is called the *trefoil*, and it has three crossings (see figure 1). The Hopf link and trefoil are examples of *alternating* links, or links in which crossings alternate.

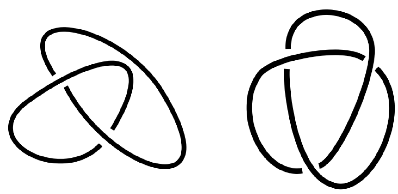


Figure 1: A Hopf link and a trefoil.

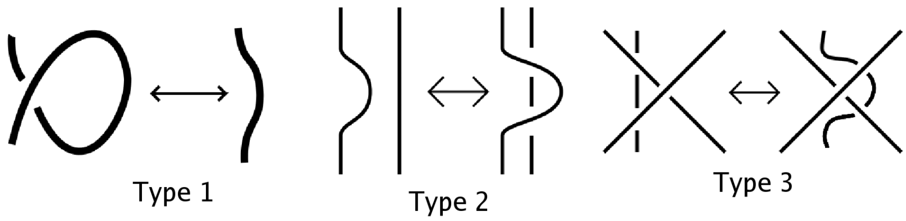


Figure 2: The Reidmeister moves.

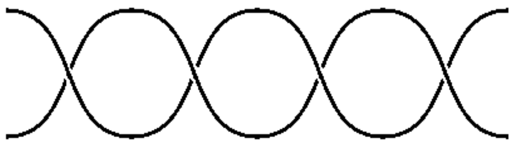


Figure 3: A four-crossing twist.

Much of Knot theory is devoted to finding ways that can tell if two different link *projections*, or pictures of links, represent different links or the same link. If two links are topologically identical, they are called *ambient isotopic*. If we have two different projections of a specific link, it has been proven that there is a sequence of *Reidemeister moves* (see figure 2) from one to the other. In more intuitive terms this means that two links are equivalent if one can be deformed into the other without tearing or allowing strands to pass through other strands.

A *twist* or an *integral tangle* in a link, is a section where two strands tangle around themselves one or more times, as seen in figure 3. The minimum number of twists taken over all projections of a link L is the *twist number* of that link and is denoted $T(L)$. Figure 5 provides an example of a knot with twist number 2. For further discussion on twists see Lin [3, p. 7].

1.1 The Bracket Polynomial

The following is a brief overview of the Bracket polynomial. For a complete explanation see Kauffman [4, p. 395-402].

One important method used to distinguish links is to associate a particular polynomial to a link projection, and hopefully every projection of a link will yield the same polynomial. One of the most successful polynomial invariants for links is the Kauffman Bracket polynomial. When computing this polynomial there are three simple rules to follow:

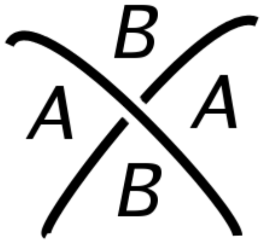
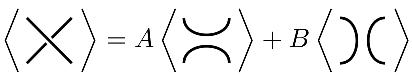
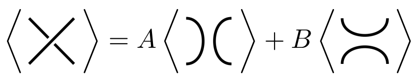


Figure 4: Crossing labels

1. $\langle O \rangle = 1$

2. $\langle O \cup L \rangle = d \langle L \rangle$

3.  *or*



Where L is an unoriented link (or knot) diagram and O represents the *unknot* which any knot ambient isotopic to a knot without crossings. Here $< L >$ is an element of the ring $\mathbb{Z}[A, B, d]$. These rules show us that: the standard projection of the unknot is assigned to the polynomial 1, when you have the standard projection of the unknot unioned though not interlinked with a link then this union is assigned to $d < L >$, and finally when you come across a crossing you can break it up into what are called an $A - smoothings$ and a $B - smoothings$.

There is a certain method of labeling which side is A and which side is B. Figure 4 shows that the A region is always to the left as you approach the crossing on the understrand and thus those regions that are not A regions are B regions. A note to the reader: usually A regions are shaded and B regions are unshaded in diagrams. Since each crossing has two different *states* we can see that there will be 2^n different ways to smooth each diagram for an n-crossing link. We then need to sum up all the states to get an intermediate polynomial.

The polynomial described so far is dependent upon the particular projection of a link; we thus need to add some changes to have a polynomial that is invariant under Reidemeister moves type two and type one as explained in [1, p.145-155]. First, in order to be invariant under Reidemeister move type two, the B's should be replaced with A^{-1}. This result yields the Kauffman bracket polynomial. Also, to be invariant under Reidemeister move type one or nugatory tangles, we need to multiply the bracket polynomial of a link with $(-A^3)^{-w(L)}$ where $w(L)$ is the writhe of the link. Finally, we take this result and replace

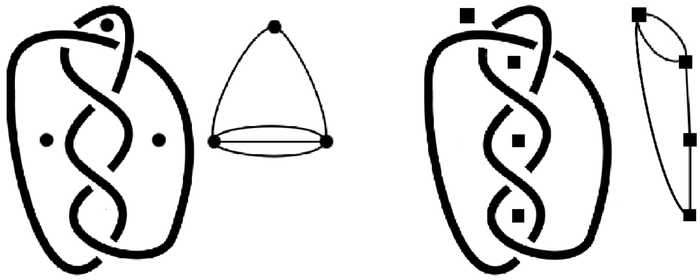
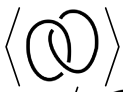


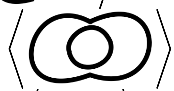
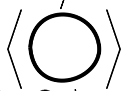
Figure 5: Retrieving planar graphs G and G', respectively, from L

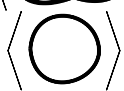
each A with a $t^{-1/4}$ to get the Jones polynomial.

As an example we will now compute the bracket polynomial of the Hopf Link:



= A  $+ A^{-1}$ 

= A [A  $+ A^{-1}$ ]

+ A^{-1}[A  $+ A^{-1}$ ]

= A[A(-(A$^2 + A^{-2}$)) $+ A^{-1}(1)] + A^{-1}[A(1) + A^{-1}(-(A^2 + A^{-2}))]$

= -A$^4 - A^{-4}$

1.2 Graphs of Link Diagrams

The following is a brief overview of some necessary graph theoretical background; for a more in depth information please refer to Thistlethwaite [5]. Given any link L we can get a *multigraph*, a graph that allows parallel edges, that represents L. To do this, first we consider all A-regions of L and put a dot in each of these regions. Whenever A-regions are connected by a crossing we draw a line from the dot of one A region to the dot of the other A-region. We get the planar multigraph $G = (V, E)$ by ignoring all the link information as seen in the first part of figure 5. Here G is made up of V its set of vertices and E its set of edges. Now if we do this process with B-regions we get the dual of G or G' as seen in the second part of figure 5. The number $n(2)$ is the number of multiedges in G, where $n^*(2)$ is the number of multiedges in G^*. The number tri is the number of triangles in G, where the multiedges are suppressed and are considered to be edges of multiplicity 1; analogously, tri^* is tri corresponding to $G*$.

1.3 Twist Numbers

Dasbach and Lin [3] describe a formula for the twist number of a reduced alternating *knot*. Given $V_K(t) = a_n t^n + a_{n+1} t^{n+1} + \cdots + a_m t^m$, the Jones polynomial of an alternating knot K, $T(K) = |a_{n+1}| + |a_{m-1}|$. The p^{th} twist number, denoted $T_p(K)$, is defined to be $|a_{n+p}| + |a_{m-p}|$. Notice that $T_1(K) = T(K)$. The following equation, taken from [3], gives an alternative description of the second twist number.

$$|a_{n+2}|+|a_{m-2}| = -|a_{m-1}||a_{n+1}| + \frac{T_1(K) + T_1^2(K)}{2} + n(2) + n^*(2) - tri - tri^* \quad (1)$$

We can generalize the definition of the p^{th} twist number to include links in the following way. Given $V_L(t) = a_n t^n + a_{n+1} t^{n+1} + \cdots + a_m t^m$, the Jones polynomial of an arbitrary alternating link L, $T_p(L) = |a_{n+p}| + |a_{m-p}|$.

2 The Second Twist Number

2.1 Achieving a Constant Second Order Twist Number

Theorem 2.1 *Let K and K' be two identical reduced alternating knot diagrams except for one twist t_K in K and $t_{K'}$ in K'. Let t_K be a three crossing twist and let $t_{K'}$ be a k-crossing twist where $k \geq 3$. Then the following equality holds: $T_2(K) = T_2(K')$.*

Proof When considering equation 1, if all the terms on the right side of the equal sign are constant when a twist in a knot has k-crossings where $k \geq 3$, it is clear that the second twist number will remain constant as well. We will now show that each term is constant.

The $-|a_{m-1}||a_{n+1}|$ term is constant because according [6], $a_{n+1} = r_w - 1$ and $a_{m-1} = r_s - 1$. Where r_w is a white *essential region* or a region that is not bounded by two crossings of the same twist; likewise r_s is a shaded essential region. These regions are not affected if a twist has a variable crossing number $k \geq 3$.

The $\frac{T(K)+T(K)^2}{2}$ term is constant because it is a function of the twist number, which by definition does not change when an already existing twist in a knot has k-crossings where $k \geq 3$.

The $n(2)+n^*(2)$ term is unchanged if a twist in a link has a variable crossing number $k \geq 3$. The reason for this is, whenever you already have a two crossing twist (with vertices outside the twist) this gives an edge a multiplicity of 2, adding more crossings just increases the multiplicity of that number. Also, when we are considering vertices within a twist it is obvious that the $n(2) + n^*(2)$ is still unaffected.

We see that the $-tri - tri^*$ term is constant when we look at a $k \geq 3$ crossing twist. We will start by considering a three crossing twist and fix all outside information. If we consider the regions enclosed within the twist we see

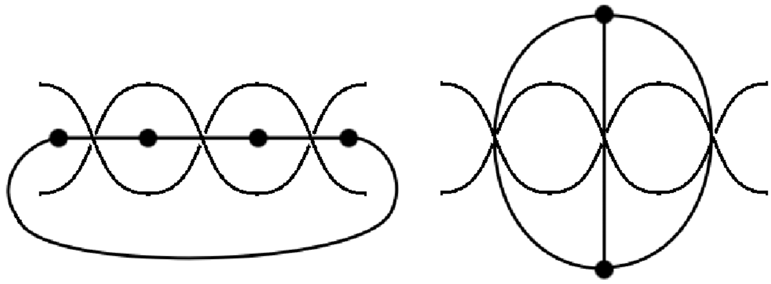


Figure 6: The values tri and tri^* are unaffected by twists with three or greater crossings.

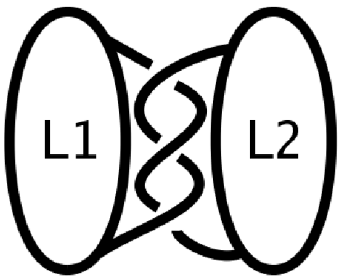


Figure 7: A link with a torus 2,3 factor, where $L1$ and $L2$ can be any link.

that there are vertices on each side of each crossing and thus there are a total of four vertices, as seen in figure 6. If we connected these vertices there is no way to yield a triangle. Thus adding more crossings (and thus vertices) will not affect the number of triangles. Similarly, if the two regions outside the twist are considered we see that there exists a multiedge with k-crossings, where k is the number of crossings in the twist. If k were three or any value larger than three, this would not affect any existing triangles since the number tri represses multiedges. ∎

2.2 Bracket Polynomial Proof

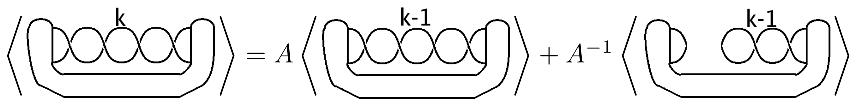
We would like to expand the last theorem to include links. However, we encounter some problems when considering links that when smoothed reduce to a link containing a shaded Hopf link factor. So the following theorem applies only when we do not come across links that have a torus 2,3 factor. A torus 2,3 factor is a part of a link that contains a three crossing twist as in figure 7.

Theorem 2.2 *Let L and L'*

1. *be two identical reduced alternating link diagrams except for one twist* t_L *in* L *and* $t_{L'}$ *in* L'
2. *not contain a torus 2,3 link factor as in figure 7, where when reduced yields a shaded Hopf link factor.*

Let t_L *be a three crossing twist and let* $t_{L'}$ *be a* k*-crossing twist where* $k \geq 3$. *Then the following equality holds:* $T_2(L) = T_2(L')$.

Proof Since the Jones polynomial is derived from the bracket polynomial, it is of interest to identify the effect of a k-crossing twist on an arbitrary link L. In particular, we will consider the second twist number coefficients.



k-crossing twist coefficients	
First:	$(-1)^{s-1}A^{-c-2s+2}$
Second:	$(-1)^{s}(r_{w,k} - h_{s,k} - 1)A^{-c-2s+6}$
Third:	$(-1)^{s-1}(t_{2,k})A^{-c-2s+10}$
Antepenultimate:	$(-1)^{w-1}(T_{2,k})A^{c+2w-10}$
Penultimate:	$(-1)^{w}(r_{s,k} - h_{w,k} - 1)A^{c+2w-6}$
Maximum:	$(-1)^{w-1}A^{c+2w-2}$

A-smoothing coefficients	
First:	$(-1)^{s}A^{-c-2s+6}$
Second:	$(-1)^{s-1}(r_{w,k-1} - h_{s,k-1} - 1)A^{-c-2s+10}$
Third:	$(-1)^{s}(t_{2,k-1})A^{-c-2s+14}$
Antepenultimate:	$(-1)^{w-1}(T_{2,k-1})A^{c+2w-10}$
Penultimate:	$(-1)^{w}(r_{s,k-1} - h_{w,k-1} - 1)A^{c+2w-6}$
Maximum:	$(-1)^{w-1}A^{c+2w-2}$

A^{-1}-smoothing coefficients	
First:	$(-1)^{s-1}A^{-c-2s+2}$
Second:	$(-1)^{s}(r_{w,k-1} - h_{s,k-1} - 2)A^{-c-2s+6}$
Third:	$(-1)^{s-1}(t_{2,0})A^{-c-2s+10}$
Antepenultimate:	$(-1)^{w+k-1}(T_{2,0})A^{c+2w-4k-10}$
Penultimate:	$(-1)^{w+k}(r_{s,k-1} - h_{w,k-1} - 1)A^{c+2w-4k-6}$
Maximum:	$(-1)^{w+k-1}A^{c+2w-4k-2}$

Here $T_{2,k}$ represents the coefficient of the antepenultimate term in the Bracket polynomial of the link with a k-crossing twist and analogously $t_{2,k}$ represents the coefficient of the third term. The exponent on the factor A is given by a the Bracket polynomial smoothing process as stated in [4]. The coefficient $r_{w,k}$ represents white essential regions in the bracket polynomial of the twist with k crossings, and thus this same k or $k-1$ notation is used in all such regions. A *hopf region* is the region bounded by a hopf link factor in a link as defined in [6]. The notation r_s signifies shaded essential regions, h_s signifies shaded hopf regions and h_w signifies shaded white hopf regions.

We see from the bracket polynomial that when $k \geq 3$, $T_{2,k} = T_{2,k-1}$ and $t_{2,k} = (r_w - h_s - 1) + t_{2,0}$. This implies that $T_{2,3} = T_{2,2}$, $T_{2,4} = T_{2,3}$, $\ldots$, $T_{2,k} = T_{2,k-1}$; and thus $T_{2,2} = T_{2,3} = T_{2,4} = \cdots = T_{2,k}$. This means that $T_{2,k}$ is constant when $k \geq 3$. Now we must show that $t_{2,k} = (r_w - h_s - 1) + t_{2,0}$ is constant for $k \geq 3$, excluding our one condition as stated in the theorem. We understand that r_w and $t_{2,0}$ is constant in L' where $t_{L'}$ has $k \geq 3$ crossings. However, the term $h_{s,k-1}$ can change only in the instance when our link L' is in the form of figure 7. The reason for this is that when one smoothing takes place, we are left with an extra Hopf link shaded region. ∎

Corollary 2.3 *Let L and L'*

1. *be two identical reduced alternating link diagrams except for one twist t_L in L and $t_{L'}$ in L'*
2. *contain exactly one torus 2,3 link factor as in figure 7, where when reduced yields a shaded Hopf link factor.*

Let t_L be a three crossing twist and let $t_{L'}$ be a k-crossing twist where $k > 3$. Then the following equality holds: $T_2(L) = T_2(L') - 1$.

Proof This corollary follows directly from the proof of Theorem 2.2.

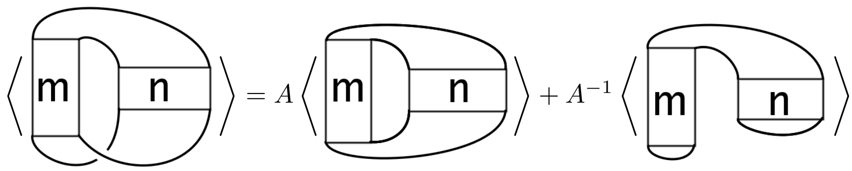
2.3 Example

As an example of this theorem, we consider an arbitrary two twist link. As below, let the boxes containing m and n represent twists of $m \geq 3$ and $n \geq 3$ crossings, respectively. Using induction, it is without loss of generality that we add one crossing to the m crossing twist; we thus show that the two twist link, with $m+1$ and n crossings, will have the same second twist number as the original two twist link (with m and n crossings). Since this case exhibits symmetry, it is clear that a link with m and $n+1$ crossing twists will also have the same second twist number as a m and n crossing twist link.

Using the $m = 3$ and $n = 3$ case as our base case we can assume that the first and last three coefficients of the m and n twist knot are as follows:

First: $(-1)^m A^{-3m-n}$
Second: $(-1)^{m-1} A^{-3m-n+4}$
Third: $2(-1)^m A^{-3m-n+8}$
Antepenultimate: $2(-1)^n A^{m+3n-8}$
Penultimate: $(-1)^{n+1} A^{m+3n-4}$
Maximum: $(-1)^n A^{m+3n}$

Note: Calculations have been omitted.



$$= (\text{-}1)^{m+1}A^{-3m-n-3} + (-1)^m A^{-3m-n+1} + 2(-1)^{m+1}A^{-3m-n+5}$$
$$+ \cdots + 2(-1)^n A^{m+3n-7} + (-1)^{n+1}A^{m+3n-3} + (-1)^n A^{m+3n+1}$$

We see that the last line is identical to the result of our assumption with $m+1$ in place of m, so we see that for all links with two twists of greater than or equal to 3 crossings we have a second twist number of 4.

3 Further Research

When Dasbach and Lin hinted at some significance of higher-order twist numbers, my goal was to find some geometric interpretation of the second order twist number. Dr. Trapp and I have considered many possibilities though most were unfruitful. The problem is therefore still open and can perhaps be generalized for the p-th twist number of reduced alternating links.

It maybe helpful to consider the second twist number as it relates to graph theory and the Tutte polynomial as seen in [2]. It may also be worthwhile to extend this idea of twist numbers to non-alternating links.

4 Acknowledgments

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